
SOME APPLICATIONS OF GENERALIZED n -METRIC SPACES

Kamran Alam Khan,

*Department of Mathematics, Ram Lubhai Sahani Govt. Mahila Degree College, Pilibhit,
(UP).*

ABSTRACT

The Generalized n -metric G_n introduced by the author extends the G -metric of Mustafa and Sims to n variables. The canonical example $G_n^d(x_1, x_2, \dots, x_n)$ equals simultaneously the *total edge length* of an $(n - 1)$ -simplex and the *total length* of the complete graph K_n whose vertices are the given points of a metric space. Building on this identification, we survey and develop some applications of Generalized n -metric in simplex geometry (volume estimates, Cayley--Menger determinants); algebraic topology (Vietoris-Rips filtrations); graph theory (complete-graph length, spanning-tree estimates); and operations research (multi-facility location, vehicle routing, and dispersion problems).

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1. Introduction

The G -metric of Mustafa and Sims [11] replaces the classical two point metric $d: X^2 \rightarrow \mathbb{R}^+$ by a fully symmetric three-point distance function $G: X^3 \rightarrow \mathbb{R}^+$ whose axioms are modelled on the perimeter of a triangle. The author [10] extended this idea to $n(\geq 3)$ points by introducing the *Generalized n -metric* $G_n: X^n \rightarrow \mathbb{R}^+$. We recall the fundamental definition of a generalized n -metric space as introduced in [10].

Definition 1.1 [10]: Let X be a non-empty set, and \mathbb{R}^+ denote the set of non-negative real numbers. Let $G_n: X^n \rightarrow \mathbb{R}^+$, $(n \geq 3)$ be a function satisfying the following properties:

[G 1]: $G_n(x_1, x_2, \dots, x_n) = 0$ if $x_1 = x_2 = \dots = x_n$

[G 2]: $G_n(x_1, x_1, \dots, x_1, x_2) > 0$ for all $x_1, x_2 \in X$ with $x_1 \neq x_2$

[G 3]: $G_n(x_1, x_1, \dots, x_1, x_2) \leq G_n(x_1, x_2, \dots, x_n)$ for all $x_1, x_2, \dots, x_n \in X$ with the condition that any two of the points x_2, \dots, x_n are distinct.

[G 4]: $G_n(x_1, x_2, \dots, x_n) = G_n(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)})$

for all $x_1, x_2, \dots, x_n \in X$ and every permutation π of $\{1, 2, \dots, n\}$

$$[G\ 5]: G_n(x_1, x_2, \dots, x_n) \leq G_n(x_1, x_{n+1}, \dots, x_{n+1}) + G_n(x_{n+1}, x_2, \dots, x_n)$$

for all $x_1, x_2, \dots, x_n, x_{n+1} \in X$

then the function G_n is called a **Generalized n -metric** on X , and the pair (X, G_n) a **Generalized n -metric space**. Two canonical examples, constructed from any metric space (X, d) are

$$G_n^d(x_1, x_2, \dots, x_n) = \sum_{1 \leq r < s \leq n} d(x_r, x_s) \dots\dots (1)$$

and

$$M_n^d(x_1, \dots, x_n) = \max_{1 \leq r < s \leq n} d(x_r, x_s) \dots\dots (2)$$

The quantity G_n^d is simultaneously the *total edge length* (perimeter of the 1-skeleton) of the $(n - 1)$ -simplex with vertices x_1, x_2, \dots, x_n , and the *total length* $L(K_n)$ of the complete graph K_n on those vertices. The quantity M_n^d is the diameter of the finite set $\{x_1, x_2, \dots, x_n\}$. This dual geometric meaning of G_n^d is the key insight that connects the Generalized n -metric to simplex geometry, graph theory, and beyond. Two fundamental inequalities from [10] will be used throughout:

$$G_n(x, y, \dots, y) \leq (n - 1)G_n(y, x, \dots, x) \dots\dots (3)$$

and

$$G_n(x_1, x_2, \dots, x_n) \leq \sum_{i=1}^{n-1} G_n(x_i, x_{i+1}, \dots, x_{i+1}) \dots\dots (4)$$

2. Applications

2.1 Simplex Geometry via Generalized n -metric

An $(n - 1)$ -simplex σ^{n-1} is the convex hull of n affinely independent points (vertices) v_0, v_1, \dots, v_{n-1} in some Euclidean space. Its 1-skeleton is the complete graph K_n formed by all $\binom{n}{2}$ edges $\{v_i, v_j\}$. The *total edge length* (or *perimeter* of the 1-skeleton) is

$$Tel(\sigma^{n-1}) = \sum_{0 \leq i < j \leq n} d(v_i, v_j) = G_n^d(v_0, v_1, \dots, v_{n-1}) \dots\dots (5)$$

For $n = 3$ this recovers the triangle perimeter $d(v_0, v_1) + d(v_1, v_2) + d(v_0, v_2)$, the motivating quantity for the G -metric [11]. The $(n - 1)$ -simplex has $\binom{n}{2}$ edges, so $Tel(\sigma^{n-1})$ is a sum of $\binom{n}{2}$ terms. Axiom [G 3] encodes a degenerate face inequality, i.e. collapsing vertices to a single point v_0 gives the least possible total edge length $G_n^d(v_0, v_0, \dots, v_0, v_1) = (n - 1)d(v_0, v_1)$, and $(n - 1)d(v_0, v_1) \leq \sum_{0 \leq i < j \leq n} d(v_i, v_j)$.

Let $d_{ij} = d(v_i, v_j)$. The squared volume of an $(n - 1)$ -simplex σ^{n-1} with vertices v_0, v_1, \dots, v_{n-1} in \mathbb{R}^m ($m \geq n - 1$) is given by

$$((n-1)!)^2 [\text{vol}(\sigma^{n-1})]^2 = \frac{(-1)^n}{2^{n-1}} \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & d_{0,1}^2 & \cdots & d_{0,n-1}^2 \\ 1 & d_{0,1}^2 & 0 & \cdots & d_{1,n-1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d_{0,n-1}^2 & d_{1,n-1}^2 & \cdots & 0 \end{vmatrix}$$

The entries of this $(n+1) \times (n+1)$ matrix are determined entirely by the pairwise distances d_{ij} , each of which is bounded above by $M_n^d(v_0, \dots, v_{n-1})$. Since $M_n^d \leq G_n^d$, bounding G_n^d directly bounds every entry and hence $\text{vol}(\sigma^{n-1})$. This makes the Generalized n -metric a natural language for metric rigidity questions in discrete geometry [3]. The classical isodiametric inequality [6] gives

$$\text{vol}(\sigma^{n-1}) \leq \frac{\omega_{n-1}}{2^{n-1}} [M_n^d(v_0, \dots, v_{n-1})]^{n-1}$$

where ω_{n-1} is the volume of the unit ball in \mathbb{R}^{n-1} .

2.2 Complete graph length and spanning tree

For vertices x_1, x_2, \dots, x_n in a metric space (X, d) , the total length of the complete graph K_n is

$$L(K_n) = \sum_{1 \leq r < s \leq n} d(x_r, x_s) = G_n^d(x_1, x_2, \dots, x_n)$$

Since every spanning subgraph H of K_n satisfies $L(H) \leq L(K_n)$, and every spanning tree T of K_n is such a subgraph, we have $L(T) \leq G_n^d$. If $MST(x_1, x_2, \dots, x_n)$ denotes the length of a minimum spanning tree on $\{x_1, x_2, \dots, x_n\}$, then

$$G_n^d(x_1, x_2, \dots, x_n) \leq (n-1)MST(x_1, x_2, \dots, x_n)$$

2.3 Vietoris--Rips and Čech complexes

Let (X, d) be a metric space and $r > 0$. The Vietoris--Rips complex $\mathcal{R}(X, r)$ has vertex set X and includes a simplex $[x_0, x_1, \dots, x_k]$ whenever $d(x_i, x_j) \leq r$ for all i, j [7]. In terms of the Generalized $(k+1)$ -metric M_{k+1}^d ,

$$[x_0, x_1, \dots, x_k] \in \mathcal{R}(X, r) \Leftrightarrow M_{k+1}^d(x_0, x_1, \dots, x_k) \leq r$$

Thus the Vietoris--Rips filtration $\{\mathcal{R}(X, r)\}_{r>0}$ is precisely the super-level filtration of M_{k+1}^d restricted to finite subsets of X . The persistent homology of this filtration [4] detects connected components, loops, and voids of X at all scales simultaneously. The diameter of the simplex $[x_0, x_1, \dots, x_k]$ is $M_{k+1}^d(x_0, x_1, \dots, x_k)$ and the two-sided estimate bounds it in terms of G_{k+1}^d :

$$M_{k+1}^d(x_0, x_1, \dots, x_k) \leq G_{k+1}^d(x_0, x_1, \dots, x_k) \leq \binom{k+1}{2} M_{k+1}^d(x_0, x_1, \dots, x_k)$$

2.4 Applications in Operations Research

2.4.1 Multi-facility location:

The k-center problem [8] places k facilities p_1, \dots, p_k in a metric space (X, d) to minimize the spread

$$S(p_1, \dots, p_k) = M_k^d(p_1, \dots, p_k) = \max_{i \neq j} d(p_i, p_j)$$

A 2-approximation algorithm is known for metric spaces [8]. Since $M_k^d \leq G_k^d$, the G_k^d formulation gives the stronger constraint that the total pairwise distance among facilities is bounded, which is relevant in diversity-aware facility placement [2].

2.4.2 Vehicle routing and TSP bounds In the Travelling Salesman Problem on n cities $x_1, \dots, x_n \in X$, the cost of any tour [1] is

$$L_{tour}(\sigma) = \sum_{i=1}^{n-1} d(x_{\sigma(i)}, x_{\sigma(i+1)}) + d(x_{\sigma(n)}, x_{\sigma(1)})$$

The Generalized n-metric G_k^d provides a universal upper bound:

$$\min_{\sigma} L_{tour}(\sigma) \leq G_n^d(x_1, \dots, x_n)$$

Christofides' 3/2-approximation for metric TSP [1] yields a corresponding lower bound.

The K-metric of [9], $K(x_1, \dots, x_n) = \frac{1}{n} [\sum_{i=1}^{n-1} d(x_i, x_{i+1}) + d(x_n, x_1)]$ is the average tour length of the polygon $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_n \rightarrow x_1$ and satisfies $K(x_1, \dots, x_n) \leq G_n^d(x_1, \dots, x_n)$

2.4.3 p-dispersion and sensor placement

The p -dispersion problem selects p points from a finite set $S \subset X$ to maximize $\delta(x_1, \dots, x_p) = \min_{i \neq j} d(x_i, x_j)$, spreading chosen locations as far apart as possible. Since

$$\delta(x_1, \dots, x_p) \leq \frac{2}{p(p-1)} G_p^d(x_1, \dots, x_p)$$

maximizing G_p^d is a sufficient condition for maximizing δ . The problem is NP-hard in general but admits a 2-approximation. Applications include antenna placement, facility siting, and experimental design.

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